SWARTHMORE COLLEGE HONORS EXAM 2024
REAL ANALYSIS

Instructions: Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

Real analysis I

1. Make the following statement precise and prove it: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Near any point $(x_0, f(x_0))$, the graph of f looks like a straight line.

2. Let (X, d) be a metric space, and let $\mathcal{K}(X)$ be the space of all nonempty compact subsets of X. We define the Hausdorff metric d_H on $\mathcal{K}(X)$ as follows: for A, $B \in \mathcal{K}(X)$, $d_H(A, B)$ is the smallest ε such that for every point a in A, there exists a point b in B with $d(a, b) \leq \varepsilon$, and for every point b in B, there exists a point a in A with $d(a, b) \leq \varepsilon$.

(a) Let $f : X \to X$ be a map. Under what conditions will f induce a continuous map $\bar{f}$ on $\mathcal{K}(X)$ (given by $\bar{f}(A) = f(A)$)?

(b) Let $\mathcal{S}$ be the set of finite sets in $\mathbb{R}$. Is $\mathcal{S}$ open in $\mathcal{K}(\mathbb{R})$ with the Hausdorff metric? Is it closed? Is it neither?

(c) Let $\{f_n : [0, 1] \to \mathbb{R} : n = 1, 2, \ldots\}$ be a sequence of continuous functions. Prove or disprove the following statement. If it's false, how can it be fixed? Statement: The functions $\{f_n\}$ converge pointwise to a function f on $[0, 1]$ if and only if the corresponding graphs, $\{(x, f_n(x)) \in \mathbb{R}^2 : x \in [0, 1]\}$, converge to the graph of f in $\mathcal{K}(\mathbb{R}^2)$ with the Hausdorff metric.

3. A sequence of non-negative real numbers a_1, $a_2, \ldots$ is called *superadditive* if $a_{m+n} \geq a_m + a_n$ for all $m, n \geq 1$. Show that for any superadditive sequence, $\displaystyle\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\displaystyle\sup_{n\to\infty} \frac{a_n}{n}$. (The limit may be $+\infty$.)

4. Let $\lfloor x \rfloor$ be the floor function, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Define a sequence of functions f_1, $f_2, \ldots : [0, 1] \to \mathbb{R}$ by $f_n(x) = \dfrac{\lfloor nx \rfloor}{n}$. Discuss the convergence of $\{f_n\}$, $\{f_n'\}$, and $\{\int_0^1 f_n(x)\, dx\}$ as $n \to \infty$. If the convergence isn't nice, explain what additional hypotheses would make it better.

5. Recall the Intermediate Value Theorem:

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and y any number between $f(a)$ and $f(b)$ inclusive. Then there exists a point $c \in [a, b]$ with $f(c) = y$.

(a) Prove the Intermediate Value Theorem.

(b) Prove the converse of the Intermediate Value Theorem, or give a counterexample showing that it's false.

6. Let $M_{2\times 2}$ be the set of real 2×2 matrices. We can give it a manifold structure by identifying it with $\mathbb{R}^4$.

(a) Is the set of 2×2, singular, upper triangular matrices a submanifold of $M_{2\times 2}$?

(b) Is the set of 2×2 matrices A such that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$ a submanifold of $M_{2\times 2}$?

7. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions, and $\int$ the Lebesgue integral. Show that

$$\sum_{n=1}^{\infty} \left(\int f_n \, d\mu \right) = \int \left(\sum_{n=1}^{\infty} f_n \right) d\mu.$$

8. Let ω be an exact two-form on $\mathbb{R}^3$. Show that there exists a one-form η of the form $\eta = f\, dx + g\, dy$ such that $d\eta = \omega$. What does this result mean in the context of vector fields?

9. Verify Stokes' theorem for the 1-form $\alpha = \dfrac{x\, dy - y\, dx}{\sqrt{x^2 + y^2}}$ on the annulus $1 \leq x^2 + y^2 \leq 2$.

10. Consider the system of equations

$$y^2 + 2u^2 + v^2 - xy = 15, \qquad 2y^2 + u^2 + v^2 + xy = 38.$$

(a) Under what conditions will this system define x and y as functions of u and v?

(b) The point $x = 1$, $y = 4$, $u = 1$, and $v = -1$ is a solution to the system. If u decreases to 0.9 and v decreases to -1.1, approximately what must happen to x and y in order for the equations to continue to hold?