Honors Exam in Topology
Swarthmore College

1) Let $p : \widetilde{X} \to X$ be a covering map, and suppose that X is path connected and has a basis of path connected sets. Let Y be a path component of $\widetilde{X}$.

a) Prove that Y is open in $\widetilde{X}$.
b) Prove that $p|Y : Y \to X$ is onto.
c) Prove that $p|Y : Y \to X$ is a covering map.

2) Let X be a set with topology F_X and let $Y = X \cup \{p\}$ where $p \notin X$. Let F_Y consist of all the sets in F_X together with all sets of the form $V \cup \{p\}$ such that $V \subseteq X$ and $X - V$ is compact and closed in X. Then F_Y is a topology for Y (but you do not have to prove this).

a) Prove that (X, F_X) is a subspace of (Y, F_Y)
b) Prove that (Y, F_Y) is compact.

3) For each $i \in I$, let X_i be a topological space and let A_i be a closed subset of X_i. Prove that $\prod_{i \in I} A_i$ is closed in $\prod_{i \in I} X_i$ with the product topology.

4) Consider $\mathbb{R}$ with the usual toplogy. Define an equivalence relation $\sim$ on $\mathbb{R}$ by declaring $x \sim y$ iff either $x = y$ or both $|x| < 1$ and $|y| < 1$. Let $X = \mathbb{R}/\sim$ with the quotient topology. Determine whether or not $X \cong \mathbb{R}$, and prove your conclusion.

5) Let I be the unit interval.
a) Let $h : I \to I$ be continuous such that $h(0) = 0$ and $h(1) = 1$. Prove that h is path homotopic to the path $f(s) = s$ on I.

b) Let X be a Hausdorff space, and let f and g be one-to-one paths in X such that $f(I) = g(I)$ and $f(0) = g(0)$ and $f(1) = g(1)$. Prove that f is path homotopic to g.

6) Let X be a path connected space and let $x, y \in X$. Let $u_f : \pi_1(X, x) \to \pi_1(X, y)$ be the isomorphism determined by a path f from x to y. Prove that u_f is independent of the particular path f if and only if $\pi_1(X, x)$ is abelian.