Swarthmore College
Department of Mathematics and Statistics
Honors Exam in Geometry 2024

Work on at least one problem from each section and complete as many questions as you can. There is no expectation that you complete all the problems.

1 Curves

1. Let $\bar{\alpha} : I \to \mathbb{R}^2$ be a unit-speed curve. Let $\{\bar{t}(s), \bar{n}(s)\}$ be the positively oriented tangent-normal basis to $\bar{\alpha}(s)$ (i.e. $\bar{n}(s)$ is $90°$ counter-clockwise from $\bar{t}(s)$), and let $\kappa(s)$ be the signed curvature at $\alpha(s)$ (i.e. $\kappa(s)$ is positive if $\bar{t}'(s)$ is a positive scalar multiple of $\bar{n}(s)$ and negative otherwise). Assume that $\kappa(s) \neq 0$ for all $s \in I$. In this situation, the curve
$$\bar{\beta}(s) = \bar{\alpha}(s) + \frac{1}{\kappa(s)}\bar{n}(s), \quad s \in I$$
is called the *evolute* of $\bar{\alpha}$.

   Show that the tangent line at s of the evolute of $\bar{\alpha}$ is the normal line to $\bar{\alpha}$ at s.

2. Suppose that $\bar{\alpha} : I \to \mathbb{R}^3$ is a curve that is parametrized by arc length and that has nonzero torsion everywhere. Show that the knowledge of the binormal vector $\bar{b}(s)$ at all points of $\bar{\alpha}$ determines the curvature $\kappa(s)$ and the absolute value of the torsion $\tau(s)$ of $\bar{\alpha}$.

2 Surfaces

3. Let S_1 be the unit sphere centered at $(0.0, 1)$, minus the north pole, and let S_2 be the xy-plane in $\mathbb{R}^3$. Let $\phi : S_1 \to S_2$ be given by stereographic projection:
$$\phi(x, y, z) = \left(\frac{2x}{2-z}, \frac{2y}{2-z}, 0\right).$$

(Note: in this problem, we are thinking of stereographic projection as a map from one surface to another, rather than as a coordinate chart of S_1.).

(a) Find a local expression (i.e. matrix) of the differential $d\phi_{\bar{p}} : T_{\bar{p}}S_1 \to T_{\phi(\bar{p})}S_2$ at the point $\bar{p} = (0, 1, 1)$ on the equator of the sphere, using the chart $\bar{x}(u, v) = (u, \sqrt{1 - u^2 - (v)^2}, v+1)$ $(u^2+v^2 < 1)$ for the sphere and the chart $\bar{y}(s, t) = (s, t, 0)$ of the plane.

(b) For a vector $\bar{w} = (a, 0, c)$ in $T_{\bar{p}}S_1$ use your answer to part (a) to describe the effect of $d\phi_{\bar{p}}$ on $\bar{w}$. Please explain carefully and include a picture.

(c) Is ϕ a diffeomorphism? Is ϕ an isometry? Please explain.

4. For a regular surface $S \in \mathbb{R}^3$, a differentiable function $f : S \to \mathbb{R}$, and a point $\bar{p} \in S$, we can define the *gradient* of f at $\bar{p}$, denoted $(\mathrm{grad} f)(\bar{p})$, to be the unique vector in $T_{\bar{p}}S$ satisfying

$$\langle (\mathrm{grad} f)(\bar{p}), \bar{v} \rangle_{\bar{p}} = df_{\bar{p}}(\bar{v})$$

for all $\bar{v} \in T_{\bar{p}}S$.

(a) Let S be the surface in $\mathbb{R}^3$ given by

$$S = \{(0, y, z) \mid y, z \in \mathbb{R}\},$$

and consider the function $f : S \to \mathbb{R}$ given by $f(0, y, z) = 3y$. Find $(\mathrm{grad} f)(\bar{p})$ at the point $(0, 1, 2) \in S$.

(b) Show that in general, for a regular surfact S and differentiable function $f : S \to \mathbb{R}$, the gradient vector $\bar{p}$ points in the direction along the surface of greatest increase of f.

3 Curvature on Surfaces

5. Let S be a sphere of radius 5, centered at the origin in $\mathbb{R}^3$ and oriented outward. Suppose that C is the horizontal cross section of S at height 4, parametrized by $\bar{\alpha}(t) = (3\cos t, 3\sin t, 4)$. Consider the point $\bar{p} = \bar{\alpha}(0) = (3, 0, 4)$.

(a) Compute the basis $\{\bar{t}, \bar{N} \times \bar{t}, \bar{N}\}$ at $\bar{p}$. Here $\bar{t}$ denotes the unit tangent to C in the direction of $\bar{\alpha}$ and $\bar{N}$ denotes the unit normal vector to the sphere. Draw a picture that indicates the curve and this basis.

(b) Compute $\frac{D\bar{\alpha}'(0)}{dt}$. Is $\bar{\alpha}$ geodesic at $\bar{p}$?

(c) Reparametrize $\bar{\alpha}$ with respect to arc length.

(d) Compute the curvature κ, the geodesic curvature κ_g, and the normal curvature κ_n of $\bar{\alpha}$ at $\bar{p}$ directly. Confirm that $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

6. Let $\bar{u}_1$ and $\bar{u}_2$ be orthonormal tangent vectors at a point $\bar{p}$ of a regular surface S. Let $\bar{N} : S \to S^2$ denote the Gauss map from the surface S to the sphere S^2.

(a) If $d\bar{N}_{\bar{p}}(\bar{u}_1) \times d\bar{N}_{\bar{p}}(\bar{u}_2) = \bar{0}$, show that the Gaussian curvature at $\bar{p}$ is 0.

(b) What geometric information can be deduced from the equation $d\bar{N}_{\bar{p}}(\bar{u}_1) \cdot \bar{u}_2 = 0$?

4 Manifolds

7. Consider the set

$$\mathrm{SL}(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, ad - bc = 1 \right\}.$$

Observe that $\mathrm{SL}(2, \mathbb{R})$ is a level set of the determinant function

$$\det : M_2(\mathbb{R}) \to \mathbb{R},$$

where the set $M_2(\mathbb{R})$ denotes the set of 2×2 matrices with real entries. Show that $\mathrm{SL}(2, \mathbb{R})$ is a differentiable manifold. What is its dimension?

8. Let (M, g) be a Riemannian manifold, and suppose that $\bar{X}$ is a vector field of constant length on (M, g). Let ∇ be the Levi-Civita connection on (M, g). Show that $\nabla_{\bar{v}} \bar{X}$ is perpendicular to $\bar{X}$ at all points $\bar{p} \in M$ and for all elements $\bar{v} \in T_{\bar{p}}M$.

9. Let $\bar{p}$ be a point in a Riemannian manifold (M, g) and let $\bar{v}$ be a nontrivial vector in $T_{\bar{p}}M$. Suppose that $\gamma(t)$ is the geodesic emanating from $\bar{p}$ with tangent vector $\bar{v}$. Define

$$S_\varepsilon = \{\bar{q} \in M \mid d(\bar{p}, \bar{q}) = \varepsilon\},$$

where $d(\bar{p}, \bar{q})$ is the distance from $\bar{p}$ to $\bar{q}$ in M. It can be shown that for small ε, S_ε is a submanifold of M. Show that for small enough ε, $\gamma(t)$ and S_ε are perpendicular at their point of intersection.