# SWARTHMORE COLLEGE HONORS EXAM 2024
# COMPLEX ANALYSIS

**Instructions:** Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

## Real analysis I

**1.** Make the following statement precise and prove it: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Near any point $(x_0, f(x_0))$, the graph of $f$ looks like a straight line.

**2.** Let $(X, d)$ be a metric space, and let $\mathcal{K}(X)$ be the space of all nonempty compact subsets of $X$. We define the Hausdorff metric $d_H$ on $\mathcal{K}(X)$ as follows: for $A$, $B \in \mathcal{K}(X)$, $d_H(A, B)$ is the smallest $\varepsilon$ such that for every point $a$ in $A$, there exists a point $b$ in $B$ with $d(a, b) \leq \varepsilon$, and for every point $b$ in $B$, there exists a point $a$ in $A$ with $d(a, b) \leq \varepsilon$.

(a) Let $f : X \to X$ be a map. Under what conditions will $f$ induce a continuous map $\bar{f}$ on $\mathcal{K}(X)$ (given by $\bar{f}(A) = f(A)$)?

(b) Let $\mathcal{S}$ be the set of finite sets in $\mathbb{R}$. Is $\mathcal{S}$ open in $\mathcal{K}(\mathbb{R})$ with the Hausdorff metric? Is it closed? Is it neither?

(c) Let $\{f_n : [0, 1] \to \mathbb{R} : n = 1, 2, \ldots\}$ be a sequence of continuous functions. Prove or disprove the following statement. If it's false, how can it be fixed? Statement: The functions $\{f_n\}$ converge pointwise to a function $f$ on $[0, 1]$ if and only if the corresponding graphs, $\{(x, f_n(x)) \in \mathbb{R}^2 : x \in [0, 1]\}$, converge to the graph of $f$ in $\mathcal{K}(\mathbb{R}^2)$ with the Hausdorff metric.

**3.** A sequence of non-negative real numbers $a_1$, $a_2, \ldots$ is called *superadditive* if $a_{m+n} \geq a_m + a_n$ for all $m, n \geq 1$. Show that for any superadditive sequence, $\displaystyle\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\displaystyle\sup_{n\to\infty} \frac{a_n}{n}$. (The limit may be $+\infty$.)

**4.** Let $\lfloor x \rfloor$ be the floor function, where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. Define a sequence of functions $f_1$, $f_2, \ldots : [0, 1] \to \mathbb{R}$ by $f_n(x) = \dfrac{\lfloor nx \rfloor}{n}$. Discuss the convergence of $\{f_n\}$, $\{f'_n\}$, and $\{\int_0^1 f_n(x)\, dx\}$ as $n \to \infty$. If the convergence isn't nice, explain what additional hypotheses would make it better.

**5.** Recall the Intermediate Value Theorem:

*Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $y$ any number between $f(a)$ and $f(b)$ inclusive. Then there exists a point $c \in [a, b]$ with $f(c) = y$.*

(a) Prove the Intermediate Value Theorem.

(b) Prove the converse of the Intermediate Value Theorem, or give a counterexample showing that it's false.

## Complex analysis

**6.** (a) Let $u(x, y)$ be a (real) harmonic function and let $(x_0, y_0)$ be a critical point for $u$. What does the second derivative test from multivariable calculus tell us about the critical point? What does that mean for extrema of complex analytic functions? Why is that not surprising?

(b) Now assume that $u(x, y)$ is positive everywhere, and that the reciprocal $\dfrac{1}{u}$ is also harmonic. Show that $u$ is constant.

**7.** For each of the following statements, either prove it or give a counterexample.

(a) If $f$ is analytic in a domain $D$ and $\Gamma$ is a loop (closed contour) in $D$, then $\int_\Gamma f(z)\,dz = 0$.

(b) If $f$ is analytic at $z_0$, then the Taylor series for $f'$ around $z_0$ can be obtained by termwise differentiation of the Taylor series for $f$ around $z_0$ and it converges in the same disk as the series for $f$.

**8.** What can you say about an entire function whose real part is never equal to its imaginary part?

**9.** Compute the following integrals.

(a) $\displaystyle\int_\Gamma \frac{z^2 e^z}{2z + i}\,dz$, where $\Gamma$ is the unit circle oriented clockwise.

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2}\,dx.$

**10.** A point $z_0$ is a *fixed point* for a function $f$ if $f(z_0) = z_0$. Let $f(z) = -2e^z - 3$. How many fixed points does $f$ have in the left half-plane?