Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra
Spring 2024

On this exam, you should aim to complete at least two problems from each section as completely as possible. If you are satisfied with your solutions to six questions, you are encouraged to attempt the other problems.

Section 1

1. (a) Show that if $\varphi : G \to H$ is a group homomorphism, then $\text{im}(\varphi)$ is a subgroup of H.
   (b) Show that if $\varphi : G \to H$ is a group homomorphism, then $|\varphi(g)| \mid |g|$.
   (c) How many group homomorphisms are there from $D_4 \to S_3$? Explicitly describe them. (Here, D_4 is the symmetry group of the square, and S_3 is the symmetric group on three elements).

2. (a) Recall that the *center* of a group G is defined by

$$Z(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

   Prove that $Z(G)$ is a normal subgroup of G.
   (b) Prove that if $G/Z(G)$ is cyclic, then G is abelian.
   (c) Let p be prime. Prove that any group of order p^2 is abelian.

3. Thoughout this problem, assume G is a group of order 30.
   (a) Prove that G is not simple.
   (b) Prove that G has a subgroup H which is isomorphic to $\mathbb{Z}/15\mathbb{Z}$.
   (c) Prove that H is normal in G.

Section 2

4. (a) Let k be a field, and let R a ring with $1_R \neq 0_R$. Suppose $\phi : k \to R$ is nontrivial ring homomorphism. Either prove that ϕ is always injective or provide a counterexample.
   (b) Let k be a field, and let R a ring with $1_R \neq 0_R$. Suppose $\phi : R \to k$ is a nontrivial ring homomorphism. Either prove that ϕ is always surjective or provide a counterexample.
   (c) Prove that the only ring homomorphism $\phi : \mathbb{C} \to \mathbb{R}$ the trivial one ($\phi(x) = 0$ for all $x \in \mathbb{C}$).

5. Let R be a commutative ring with $1_R \neq 0_R$. We say that an ideal $I \subsetneq R$ is prime if whenever $ab \in I$ for some $a, b \in R$, we have $a \in I$ or $b \in I$.
   (a) Show that if $I_1, I_2 \subseteq R$ are ideals, then $I_1 \cap I_2 := \{a \in R \mid a \in I_1 \text{ and } a \in I_2\}$ is also an ideal of R.
   (b) Show that an ideal $I \subsetneq R$ is prime if and only if the following two conditions hold.

i. If $I = I_1 \cap I_2$ for any two ideals $I_1, I_2 \subseteq R$, then $I = I_1$ or $I = I_2$, and
ii. If $a \in R$ and $a^n \in I$ for some positive integer n, then $a \in I$.

6. Let k be a field. Prove that none of the following rings are isomorphic (you should be making three comparisons).

   (a) $k[x, y]/(y - x^2)$

   (b) $k[x, y]/(y^2 - x^2)$

   (c) $k[x, y]/(y^3 - x^2)$

Section 3

7. (a) Prove that an algebraically closed field must be infinite.

   (b) Let $\mathbb{F}_p$ denote the finite field with p elements. Prove that there are irreducible polynomials of arbitrarily high degree in $\mathbb{F}_p[x]$.

8. Throughout, denote by $\phi(n)$ the number of non-negative integers less than n that are relatively prime to n.

   (a) For any $n > 1$, describe the Galois group of $x^n - 1$ over $\mathbb{Q}$.

   (b) Let p be prime. Show that there is an integer n such that $p|\phi(n)$.

   (c) Let p be prime. Show that there exists a Galois extension K over $\mathbb{Q}$ with $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z}$.

9. Let $f(x) = x^4 + 5x^2 + 5 \in \mathbb{Q}[x]$.

   (a) Prove that f is irreducible over $\mathbb{Q}$.

   (b) Determine the Galois group of f.

   (c) Determine all subfields of the splitting field of f.