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complex conjugates. As R increases to R_b, the roots move along a curved path to points s_b. The radial distance from the origin to either root is given by

$$\sqrt{\alpha^2 + \omega^2} = \sqrt{\left(-\frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} = \sqrt{\frac{1}{LC}} = \omega_n \qquad (4\text{-}24)$$

a constant. Therefore, the locus is a circular arc of radius ω_n.

The critical value of resistance is defined by $\omega = 0$ or, in other words, where the discriminant equals zero. For this condition $\sigma = -\alpha_c = -R/2L = -\sqrt{1/LC}$. At this value of $R = R_c$, the roots coincide on the real axis. As R increases to R_d, s_{d_1} moves toward the origin and s_{d_2} moves out along the negative real axis. As R increases without limit, s_1 approaches the origin and s_2 increases without limit.

Exercise 4-8

For the series RLC circuit of Example 3, let $L = 10$ mH and $C = 1\ \mu$F. Determine:

(a) The undamped natural frequency ω_n.
(b) The critical value of the resistance R_c and the critical damped roots $s_c = s_1 = s_2$.

Answers: (a) $\omega_n = 10^4$ rad/s; (b) $R = 200\ \Omega$; and $s_c = -10^4\ s^{-1}$.

With a different circuit parameter the locus takes on a different shape (see Problem 4-23). In any such plot, called a *root locus,* the location of the roots determines the character of the natural response, and the effect of changes in a parameter is clearly visible to the initiated. The first step in developing this powerful tool is mastery of a new concept—*impedance.*

4-4 IMPEDANCE CONCEPTS

If we limit the discussion to exponential waveforms, some interesting and valuable relations between voltage and current can be established. Actually this is not

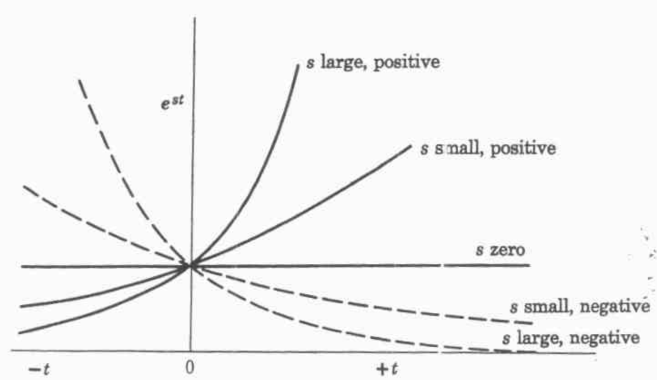


Figure 4.12 The range of exponential functions for real values of s.

a severe restriction; already we have used exponentials to represent sinusoids and exponentials with $s = 0$ can be used to represent direct currents. The range of exponential functions for positive and negative, large and small, real values of s is indicated in Fig. 4.12.

The key property of an exponential function is that its time derivative is also an exponential. For example, if

$$i = I_O e^{st}, \qquad \frac{di}{dt} = sI_O e^{st} = si \tag{4-25}$$

or if

$$v = V_O e^{st}, \qquad \frac{dv}{dt} = sV_O e^{st} = sv \tag{4-26}$$

This property greatly facilitates calculating the response of circuits containing resistance, inductance, and capacitance because of the simple voltage–current relations that result.

IMPEDANCE TO EXPONENTIALS

The ratio of voltage to current for exponential waveforms is defined as the *impedance Z*. For a resistance, $v = Ri$ and

$$Z_R = \frac{v}{i} = \frac{Ri}{i} = R \text{ in ohms} \tag{4-27}$$

For an inductance, $v = L(di/dt) = sLi$ and

$$Z_L = \frac{v}{i} = \frac{sLi}{i} = sL \text{ in ohms} \tag{4-28}$$

For a capacitance, $i = C(dv/dt) = sCv$ and

$$Z_C = \frac{v}{i} = \frac{v}{sCv} = \frac{1}{sC} \text{ in ohms} \tag{4-29}$$

The impedances Z_R, Z_L, and Z_C are constants of proportionality between voltages and currents that are exponential functions of time t and frequency s. It can be demonstrated that in each case the dimensions of impedance are the same as those of resistance. (Can you do it?) The general relation

$$v = Zi \tag{4-30}$$

corresponds to Ohm's law for purely resistive circuits, but it must be emphasized that impedance is defined *only for exponentials* and for waveforms that can be represented by exponentials.† With this restriction, impedances can be combined in series and parallel just as resistances are, and network theorems can be extended to include circuits containing L and C.

† Assume another waveform such as $i = I_O t$; then $v_L = L(di/dt) = LI_O$ and the ratio $v_L/i = LI_O/I_O t = L/t$ is a function of time and *not* a constant as it is for exponentials.

Example 5

In the circuit of Fig. 4.13, $R_1 = 2\ \Omega$, $C = 0.25$ F, and $R = 4\ \Omega$. Using the impedance concept, find the currents i and i_C for a voltage $v = 6\ e^{-2t}$ V.

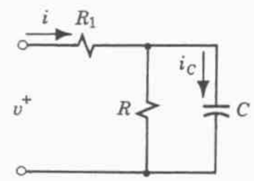


Figure 4.13 The impedance concept in circuit analysis.

Following the rules for resistive networks,

$$Z = Z_1 + \frac{Z_R Z_C}{Z_R + Z_C} = R_1 + \frac{R(1/sC)}{R + (1/sC)}$$

$$= 2 + \frac{4/(-2 \times 0.25)}{4 + 1/(-2 \times 0.25)} = 2 + \frac{-8}{2} = -2\ \Omega$$

Then

$$i = \frac{v}{Z} = \frac{6\ e^{-2t}}{-2} = -3\ e^{-2t}\ \text{A}$$

Using the current-divider and impedance concepts,

$$i_C = \frac{Z_R \cdot i}{Z_R + Z_C} = \frac{4 \cdot i}{4 - 2} = -6\ e^{-2t}\ \text{A}$$

THE IMPEDANCE FUNCTION

For the circuit of Fig. 4.14, the governing equation is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\ dt = v \tag{4-31}$$

For an exponential current $i = I_O\ e^{st}$, this becomes

$$Lsi + Ri + \frac{1}{sC}i = v$$

and

$$Z = \frac{v}{i} = sL + R + \frac{1}{sC} \tag{4-32}$$

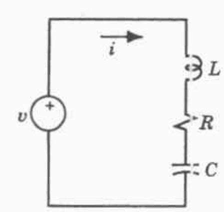


Figure 4.14 The impedance of an *RLC* circuit.

Note that when the characteristic equation (Eq. 4-13) is divided through by s, the right-hand side of Eq. 4-32 appears; but Eq. 4-32 was obtained by expressing the impedance as a ratio of an exponential voltage to an exponential current. The same relation could also have been obtained by considering that the resultant of impedances in series is the algebraic sum of the impedances or

$$Z = Z_L + Z_R + Z_C = Z(s)$$

where $Z(s)$ signifies the impedance to exponentials of the form $A\ e^{st}$.

Because the impedance function $Z(s)$ contains the same information as the characteristic equation, it is a useful concept in predicting the natural behavior of a system and it can be extended to include the prediction of steady-state response.

Example 6

Given the circuit of Example 5 with $v = V_O e^{st}$, determine and plot $Z(s)$ for real values of s.

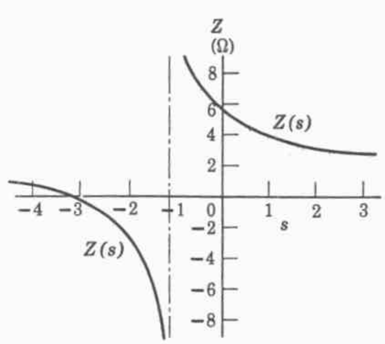


Figure 4.15 The impedance function for the circuit of Fig. 4.13.

A great advantage of the impedance concept is the ease with which impedances may be combined. Here

$$Z(s) = Z_1 + \frac{Z_R Z_C}{Z_R + Z_C} = R_1 + \frac{R(1/sC)}{R + 1/sC}$$

$$= R_1 + \frac{R}{RsC + 1} = \frac{sR_1RC + R_1 + R}{sRC + 1}$$

For $R_1 = 2\ \Omega$, $R = 4\ \Omega$, and $C = 0.25$ F,

$$Z(s) = \frac{2s + 6}{s + 1} = 2\frac{s + 3}{s + 1}$$

From the graph of Fig. 4.15 and from the expression for $Z(s)$ it is seen that $Z(s) = 0$ at $s = -3$ and that $Z(s)$ increases without limit as s approaches -1.

As indicated in Example 6, the impedance function can be obtained easily in a circuit for which the governing equation may be quite complicated (i.e., it may consist of a set of simultaneous integrodifferential equations).

Exercise 4-9

Consider the circuit shown in Fig. E4.9 where $i = e^{-5t}$A, $L = 1$ H, $C = 1$ F, and $R = 1\ \Omega$. Find the impedance Z of the circuit and v and i_R. The arrow with Z implies that we desire the impedance looking to the right from the source.

Answers: $Z = -\frac{5}{21}\ \Omega$, $v = -\frac{5}{21} e^{-5t}$ V, $i_R = -\frac{5}{21} e^{-5t}$ A.

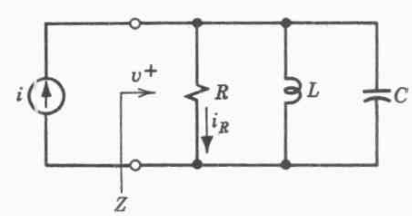


Figure E4.9

Exercise 4-10

For the circuit of Fig. 4.13 with $R_1 = 1\ \Omega$, $R = 1\ \Omega$, and $C = 0.5$ F, determine and plot $Z(s)$ for real values of s.

Answer: $Z(s)$ intersects the x-axis at -4 and intersects the y-axis at $+2$.

4-5 POLES AND ZEROS

In Example 6, $Z(s)$ is plotted for real values of s. Since each value of s corresponds to a particular exponential function, the impedance $Z(s_1)$ is the ratio of voltage to current for the particular exponential function of time $i = I_O e^{s_1 t}$. For instance, to determine the opposition to direct current flow, the impedance is

Example 7

Plot the magnitude of the impedance function of the circuit of Example 5 for real values of s and for imaginary values of s, and then sketch the impedance surface.

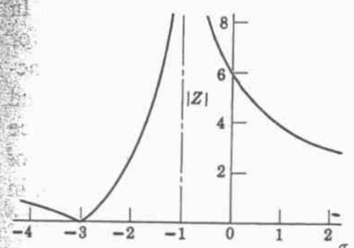
The magnitude of $Z(s)$ for real values of s corresponds to the graph of Fig. 4.15 with negative quantities plotted above the axis. For imaginary values of $s(\pm j1,\ \pm j2$, etc.) the magnitude of $Z(s)$ is calculated and plotted in a similar way. For $s = -j$,

$$Z(s) = 2\,\frac{3-j}{1-j} \quad \text{and} \quad |Z| = \frac{2\sqrt{10}}{\sqrt{2}} \cong 4.5\ \Omega$$

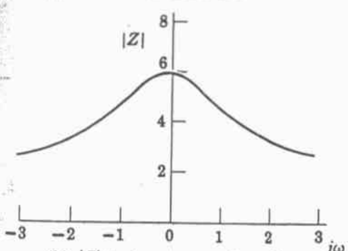
For the complex value $s = -1 - j2$,

$$Z(s) = 2\,\frac{2-j2}{0-j2} \quad \text{and} \quad |Z| = \frac{2\sqrt{8}}{\sqrt{4}} \cong 2.8\ \Omega$$

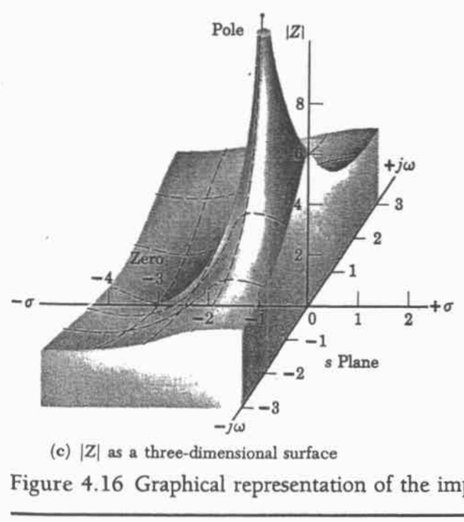
Using the profiles of Fig. 4.16a and b and additional points in the complex plane, the impedance surface is sketched in Fig. 4.16c. The points of zero and infinite impedance are plotted in Fig. 4.16d.



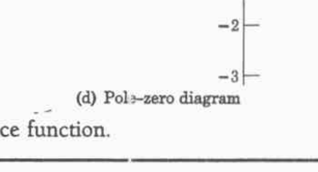
(a) |Z| for real values of s



(b) |Z| for imaginary values of s



(c) |Z| as a three-dimensional surface



(d) Pole-zero diagram

Figure 4.16 Graphical representation of the impedance function.

calculated for $s = 0$ since $i = I_O e^{(0)t} = I_O$, a direct current (dc). In Example 6, $Z(0) = 6\ \Omega$, the equivalent of the two resistances in series. This is physically correct (Fig. 4.13) because under steady conditions with no change in the voltage across the capacitor, no current flows in C and the circuit, in effect, consists only of the two resistors.

Every point in the complex frequency plane defines an exponential function and the complete plane represents all such functions. The magnitude of the impedance, $|Z(s)|$, can be plotted as a vertical distance above the s plane and, in general, the result will be a complicated surface. (See Example 7 on p. 129.)

Typically, the impedance function in three dimensions has the appearance of a tent pitched on the s plane. The height of the tent (magnitude of Z) becomes very great (approaches infinity) for particular values of s appropriately called *poles*. The tent touches the ground at particular values of s called *zeros*. The equation of the surface is usually quite complicated, but the practical use of the impedance function in network analysis or synthesis is relatively simple for two reasons. In the first place, we are usually interested in only a single profile of the surface; for example, the profile along the $j\omega$ axis indicates the response to sinusoidal functions of various frequencies. Second, just as two points define a straight line and three points define a circle, the poles and zeros uniquely define the impedance function except for a constant scale factor. As a result, the *pole-zero diagram* of Fig. 4.16d contains the essential information of the impedance function shown in Fig. 4.16c. The locations of the poles (X) and zeros (O), which are relatively easy to find, tell us a great deal about the natural response.

Exercise 4-11

(a) Plot $|Z|$ of Exercise 4-10 for imaginary values of s.
(b) Plot the pole–zero diagram of $Z(s)$ of Exercise 4-10.

Answer: (a) $|Z|$ is equal to 2 at $s = 0$ and 1.58 at $s = j2$; (b) a pole at -2 and a zero at -4.

PHYSICAL INTERPRETATION OF POLES AND ZEROS

In the general relation between exponential voltage and current

$$v = Zi$$

what is the meaning of "zero impedance"? If the impedance is small, a given current can exist with a small voltage applied. Carrying this idea to the limit, under the conditions of zero impedance† a current can exist with *no applied*

† To get the "feel" of zero impedance in a physical system, perform the following mental experiment. Grasp the bob of an imaginary pendulum between your thumb and forefinger. Applying a force, cause the bob to move along its arc with various motions. For motions against gravity or against inertia, appreciable force is required; the mechanical impedance is appreciable. To maintain a purely sinusoidal motion at the natural frequency of the pendulum only a little force (just that needed to make up friction losses) is required; the mechanical impedance to such a motion is small. To maintain a motion approximating an exponentially decaying sinusoid of just the right frequency, *no* force is required; for this motion, velocity is possible with no applied force. In other words, to this motion mechanical impedance is *zero*!

voltage; but a current with no applied voltage is, by definition, a natural current response. We conclude, therefore, that each zero, $s = s_1$, of the impedance function for any circuit designates a possible component, $I_1 e^{s_1 t}$, of the natural response current of that circuit. This conclusion is supported by the fact that for the circuit of Fig. 4.14, the zeros of the impedance function (Eq. 4-32) are identical with the roots of the characteristic equation (Eq. 4-13). Knowing the locations of the impedance zeros, we can immediately identify the exponential components of the natural current.

Example 8

Given the circuit shown in Fig. 4.17, determine the poles and zeros of impedance. If energy is stored in the circuit in the form of an initial voltage V_O on the capacitor, predict the current i that will flow when the switch S is closed.

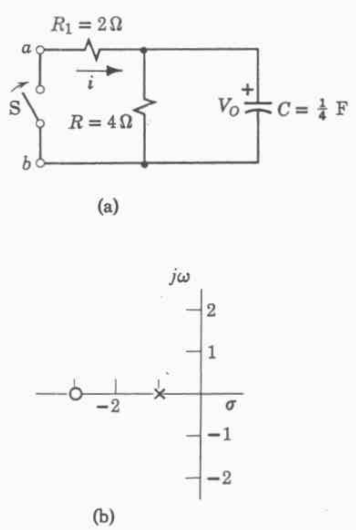


Figure 4.17 A pole-zero diagram of impedance.

As previously determined, the impedance function looking into the circuit at terminals ab is

$$Z(s) = 2\,\frac{s+3}{s+1}$$

Where $s = -1$, the denominator is zero and
$Z(s) = \infty$; therefore,
$s = -1$ is a pole.
Where $s = -3$, the numerator is zero and
$Z(s) = 0$; therefore,
$s = -3$ is a zero.

If the impedance is zero, a current can exist with no external forcing voltage. Therefore, the natural current behavior is defined by $s = s_1 = -3$ or

$$i = I_1 e^{-3t}$$

As before, I_1 is evaluated from initial data. At the instant the switch is closed, V_O appears across R_1 (tending to cause a current opposite to that assumed) and

$$i_O = I_1 e^0 = I_1 = -\frac{V_O}{R_1}$$

Hence

$$i = -\frac{V_O}{R_1} e^{-3t}$$

is the natural response current.

Exercise 4-12

Determine the current through and the voltage across the capacitor C of the circuit of Example 8 for $t > 0$. Use the current-divider principle.

Answer: $v = V_O e^{-3t}$ V, $i = -0.75\, V_O e^{-3t}$ A.

If an impedance zero indicates the possibility of a current without a voltage, what is the significance of an impedance pole? Since $v = Zi$, if the impedance is very large, a given voltage can exist with only a small current flowing. Carrying this idea to the limit, under conditions of infinite impedance a voltage can exist with *no current flow,* in other words, a natural voltage. We conclude that each pole, $s = s_a$, of the impedance function of a circuit designates a possible component, $V_a e^{s_a t}$, of the natural voltage response of that circuit.

Example 9

Returning to Example 8, assume that energy is stored in the form of an initial voltage V_O on the capacitor (perhaps by means of the voltage source shown in Fig. 4.18). Predict the votage v that will appear across terminals ab when the switch S is opened.

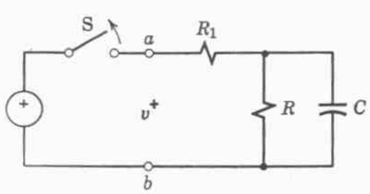


Figure 4.18 Natural voltage response.

With the external energy source removed, only a natural behavior voltage can appear. Such a voltage can exist with no current flow only if the impedance is infinite. For a pole at $s = s_a = -1$, the natural voltage behavior is

$$v = V_a e^{-t}$$

V_a is evaluated from initial data. At the instant the switch is opened, the current in R_1 goes to zero and the voltage across terminals ab is just $v_R = v_C = + V_O$. Hence,

$$v = V_O e^{-t}$$

is the natural response voltage. This result can be checked by letting $v = i_2 R$ where i_2 is the natural current that would flow in R if R were suddenly connected across the charged capacitor.

Exercise 4-13

Determine the equivalent resistance R_{eq} for each of the circuits in: (a) Example 8 and (b) Example 9 for $t > 0$. *Hint:* Review the discussion of the circuits shown in Fig. 4.6. Note that the resistance R_1 does not enter into the calculation in part (b) after the switch is open.

Answers: (a) $R_{eq} = 1.333\ \Omega$; (b) $R_{eq} = 4.0\ \Omega$.

THE GENERAL IMPEDANCE FUNCTION

Typical impedance functions include (for the circuit of Example 6)

$$Z(s) = R_1 + \frac{R(1/sC)}{R + (1/sC)} = \frac{sR_1RC + R_1 + R}{sRC + 1}$$

and (for the circuit of Fig. 4.14)

$$Z(s) = sL + R + \frac{1}{sC} = \frac{s^2LC + sRC + 1}{sC}$$

The impedance function for any network, no matter how complicated, consisting of resistances, inductances, and capacitances, can be reduced to the ratio of two polynomials in s. In general, we can write

$$Z(s) = K\frac{s^n + \cdots + k_2s^2 + k_1s + k_0}{s^m + \cdots + c_2s^2 + c_1s + c_0} \tag{4-33}$$

Although it may not be easy, this can always be factored into

$$Z(s) = K\frac{(s - s_1)(s - s_2) \cdots (s - s_n)}{(s - s_a)(s - s_b) \cdots (s - s_m)} \tag{4-34}$$

When $s = s_1, s_2, \ldots, s_n$, $Z(s) = 0$; therefore, these are zeros.
When $s = s_a, s_b, \ldots, s_m$, $Z(s) = \infty$; therefore, these are poles.

If the network is known, the impedance function can be written and factored and the pole–zero diagram constructed. Conversely, from the pole–zero diagram the impedance function can be formulated (except for scale factor K in Eq. 4-34). Theoretically, a network can then be designed or "synthesized," but practically this is not always easy. The complicated problems are so difficult that entire books have been written on the subject of network synthesis. In this book, however, we are concerned primarily with network analysis and the problems are more straightforward.

Exercise 4-14

A circuit has an impedance

$$Z(s) = \frac{(s^2 + 4s + 8)(s + 1)}{s^2 + 2s + 5}$$

Determine and plot the poles and zeros of $Z(s)$ in the complex s plane.

Answer: Poles: $-1 + j2$, $-1 - j2$; zeros: -1, $-2 + j2$, $-2 - j2$.

For a physical circuit or network, all terms and coefficients of the resulting impedance $Z(s)$ are real. Therefore, if there are complex poles or zeros, these poles or zeros must occur in complex conjugate pairs. A similar conclusion can be drawn for the admittance $Y(s)$ as discussed below.

THE GENERAL ADMITTANCE FUNCTION

Impedance is defined for exponentials as the ratio of voltage to current. The reciprocal of impedance is *admittance*, a useful property defined as the ratio of exponential current in amperes to voltage in volts so that

$$Y = \frac{i}{v} = \frac{1}{Z} \tag{4-35}$$

measured in siemens. For an ideal resistance, the admittance is just the conductance or

$$Y_R = \frac{i_R}{v_R} = \frac{i}{Ri} = \frac{1}{R} = G$$

For exponential voltages and currents, the admittances of ideal inductive and capacitive circuit elements are

$$Y_L = \frac{i_L}{v_L} = \frac{i}{L(di/dt)} = \frac{i}{sLi} = \frac{1}{sL} \qquad (4\text{-}36)$$

$$Y_C = \frac{i_C}{v_C} = \frac{C(dv/dt)}{v} = \frac{sCv}{v} = sC$$

Admittance is particularly useful in analyzing circuits that contain elements connected in parallel. Since current is directly proportional to admittance ($i = Yv$), admittances in parallel can be added directly just as conductances in parallel are added. For example, the total admittance of a parallel GCL circuit is

$$Y_L(s) = G + sC + 1/sL$$

The admittance function $Y(s)$ for any network consisting of lumped passive elements also can be reduced to the ratio of two polynomials in s. In the standard factored form this becomes

$$Y(s) = \frac{1}{Z(s)} = \frac{1}{K}\frac{(s - s_a)(s - s_b) \cdots (s - s_m)}{(s - s_1)(s - s_2) \cdots (s - s_n)} \qquad (4\text{-}37)$$

Note that the admittance function has poles where the impedance function has zeros, and vice versa. The pole-zero diagram for the admittance function contains the same information as the pole-zero diagram for the impedance function, but the diagrams are labeled differently. (See Exercise 4-15 on p. 136.)

GENERAL PROCEDURE FOR USING POLES AND ZEROS

The pole–zero concept is a powerful tool in determining the natural behavior (and, as we shall see in Chapter 5, the forced behavior) of any linear physical system. Modified to take advantage of this concept, the general procedure for determining the natural behavior of an electrical circuit is:

1. Write the impedance or admittance function for the terminals of interest.
2. Determine the poles and zeros, and plot the pole–zero diagram.
3. (a) For the terminals short-circuited, the natural behavior current is

$$i = I_1 e^{s_1 t} + I_2 e^{s_2 t} + \cdots + I_n e^{s_n t} \qquad (4\text{-}38)$$

   where $s_1, s_2, \ldots, s_n$ are zeros of the impedance function or poles of the admittance function.

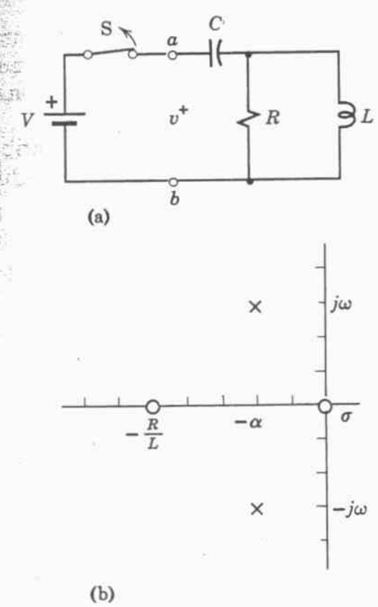
   (b) For the terminals open-circuited, the natural behavior voltage is

$$v = V_a e^{s_a t} + V_b e^{s_b t} + \cdots + V_m e^{s_m t} \qquad (4\text{-}39)$$

   where $s_a, s_b, \ldots, s_m$ are poles of the impedance function or zeros of the admittance function.
4. Evaluate the coefficients from the initial conditions (Example 10).

Example 10

(a) In the circuit of Fig. 4.19a, the voltage source has been connected for a time long enough for steady-state conditions to be reached. At time $t = 0$, switch S is opened. Predict the open-circuit voltage across terminals ab.



(b) Derive an expression for current i_R if the terminals ab are short-circuited by means of switch S_2 as shown in Fig. 4.19c.

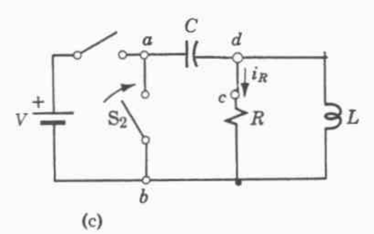


Figure 4.19 The use of poles and zeros.

Following the general procedure,

1. $$Z_{ab}(s) = \frac{1}{sC} + \frac{RsL}{R + sL} = \frac{R + sL + s^2RLC}{sC(R + sL)}$$
$$= R\,\frac{s^2 + (1/RC)s + 1/LC}{s(s + R/L)} = \frac{1}{Y(s)}$$

In the general form of Eq. 4-37,

$$Y_{ab}(s) = \frac{1}{R}\,\frac{(s - 0)(s - [-R/L])}{(s - s_1)(s - s_2)}$$

2. The admittance function has zeros at $s = 0$ and $s = -R/L$, and poles at $s = s_1$ and $s = s_2$ where

$$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}$$

Assuming complex poles, the pole–zero diagram of admittance is as shown in Fig. 4.19b.

3. For terminals ab open-circuited, the natural behavior voltage is defined by the zeros of $Y(s)$ or

$$v = V_a\,e^0 + V_b\,e^{-(R/L)t}$$

4. At $t = 0^-$, the current in the inductance is constant and $v_L = L\,di/dt = 0 = v_R$, $\therefore$ all the voltage V appears across the capacitance and the current in the inductance is zero. The second component of voltage (which reflects the *possibility* of energy storage in the inductance) is zero. The only energy storage is in the capacitor; at $t = 0^+$,

$$v = V_a\,e^0 = V$$

or the open-circuit voltage across terminals ab is just V. (There is no way for this ideal capacitor to discharge.)

We interpret the current of interest, i_R, as a short-circuit current, in this case at terminals cd. To this current the impedance is

$$Z_{cd}(s) = R + \frac{sL/sC}{sL + 1/sC} = \frac{s^2RLC + sL + R}{s^2LC + 1}$$

For $R = 1\ \Omega$, $L = 0.2$ H, and $C = 0.1$ F,

$$Z_{cd}(s) = \frac{0.02s^2 + 0.2s + 1}{0.02s^2 + 1} = \frac{s^2 + 10s + 50}{s^2 + 50}$$

The zeros are $s_1, s_2 = -5 \pm j5$. Therefore,

$$i_R = I_R\,e^{-5t}\sin(5t + \theta)$$

The zeros of $Z_{cd}(s)$ and $Z_{ab}(s)$ are the same because the circuits are the same for ab short-circuited. The poles differ, however, because the open circuits are quite different.

Exercise 4-15

A circuit has an admittance

$$Y(s) = \frac{s^2 + 4s + 5}{(s^2 + 2s + 4)(s + 3)}$$

Determine and plot the poles and zeros of $Y(s)$. Compare the poles and zeros of $Y(s)$ with those of $Z(s)$ of Exercise 4-14.

Answer: Poles: -3, $-1 + j\sqrt{3}$, $-1 - j\sqrt{3}$; zeros: $-2 + j1$, $-2 - j1$.

SUMMARY

- Forced behavior is the response to external energy sources.
  Natural behavior is the response to internal stored energy.
- Many physical systems with one energy-storage element can be described adequately by first-order integrodifferential equations. The general procedure for determining the natural behavior of a linear system is:
  1. Write the governing integrodifferential equation.
  2. Reduce this to a homogeneous differential equation.
  3. Assume an exponential solution with undetermined constants.
  4. Determine the exponents from the homogeneous equation.
  5. Evaluate the coefficients from the given conditions.
- In a second-order system with two energy-storage elements, the character of the natural response is determined by the discriminant.
  If the discriminant is *positive*, the response is *overdamped* and is represented by the sum of two decaying exponentials: $a = A_1 e^{s_1 t} + A_2 e^{s_2 t}$.
  If the discriminant is negative, the response is *oscillatory* and is represented by the damped sinusoid: $a = A e^{-\alpha t} \sin(\omega t + \theta)$.
  If the discriminant is *zero*, the response is *critically damped* and is represented by the sum of two different terms: $a = A_1 e^{st} + A_2 t e^{st}$.
- Impedance Z (ohms) and admittance Y (siemens) are defined for exponentials.

  Where $v = Zi$, $Z_R = R$, $Z_L = sL$, and $Z_C = 1/sC$.
  Where $i = Yv$, $Y_R = G$, $Y_L = 1/sL$, and $Y_C = sC$.
- Impedances and admittances in complicated networks are combined in the same way as resistances and conductances, respectively.
  The impedance function $Z(s)$ and the admittance function $Y(s)$ contain the same information as the characteristic equation.
- The pole–zero diagram contains the essential information of the impedance function or the admittance function.
  A zero of the impedance function indicates the possibility of a current without an applied voltage; therefore, a natural current.
  A pole of the impedance function indicates the possibility of a voltage without an applied current; therefore, a natural voltage.

- Using the pole–zero concept, the general procedure is:
  1. Write the impedance or admittance function for the terminals of interest.
  2. Determine the poles and zeros of impedance or admittance.
  3. Use the poles and zeros to identify possible components of natural voltage or current.
  4. Evaluate the coefficients from the given conditions.

TERMS AND CONCEPTS

admittance Ratio of exponential current in amperes to voltage in volts; the reciprocal of impedance.

characteristic equation Equation derived from the governing equation for a circuit by ignoring all energy sources and assuming an exponential solution; it contains all the information necessary for determining the character of the natural response.

damping Reduction in amplitude of response with time.

discriminant Quantity under the radical sign in the solution of the second-order characteristic equation that indicates the character of the natural response.

impedance Ratio of exponential voltage in volts to current in amperes; the reciprocal of admittance.

natural response Circuit behavior, due to internal energy storage alone, that depends only on the nature of the circuit.

pole Particular value of s for which the magnitude of impedance $Z(s)$ or admittance $Y(s)$ approaches infinity.

second-order system Circuit or system for which the homogeneous differential equation contains a second-degree term due to the presence of two independent energy storage elements.

zero Particular value of s for which the magnitude of impedance $Z(s)$ or admittance $Y(s)$ goes to zero.

REVIEW QUESTIONS

1. Cite an example of natural behavior in each of the following branches of engineering: aeronautical, chemical, civil, industrial, and mechanical.
2. To what extent is the natural behavior of a system influenced by the waveform of the forcing function that stores energy in the system?
3. Discuss the possibility of a positive exponent appearing in the natural response of a passive network. In any physical system.
4. Outline the procedure for determining the natural behavior of a mechanical system in translation.
5. In contrast to the exponential behavior of its idealized model, an actual coasting automobile comes to a complete stop long before an infinite time. Why?
6. Measurements on a certain very fast transient are difficult and the results include large random errors. In determining the time constant for this system, would a linear or a semilog plot of experimental values be preferable? Why?
7. What is the physical difference between a "first-order" system and a "second-order" system?
8. In what sense is the "characteristic equation" characteristic of the circuit?
9. In what sense does the "discriminant" discriminate?
10. What important initial information is available